J. Fluid Mech. (1991), *vol.* 232, *pp.* 415–438
Printed in Great Britain

The unsteady laminar boundary layer on an axisymmetric body subject to small-amplitude fluctuations in the free-stream velocity

By PETER W. DUCK

Department of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, UK

(Received 28 June 1990 and in revised form 2 May 1991)

The effect of small-amplitude, time-periodic, free-stream disturbances on an otherwise steady axisymmetric boundary layer on a circular cylinder is considered. Numerical solutions to the problem are presented, and an asymptotic solution to the flow, valid far downstream along the axis of the cylinder is detailed. Particular emphasis is placed on the unsteady eigensolutions that occur far downstream, which turn out to be very different from the analogous planar eigensolutions. These axisymmetric eigensolutions are computed numerically and also are described by asymptotic analyses valid for low and high frequencies of oscillation.

1. Introduction

The effect of time-periodic disturbances in the free stream of an otherwise steady boundary layer has received considerable attention over the years. This work was initiated by Lighthill (1954), who considered the flow past a semi-infinite flat plate, with a small-amplitude, time-periodic, free-stream disturbance, and obtained solutions close to and far from the leading edge. This work was later extended by Rott & Rosenweig (1960), Lam & Rott (1960) and Ackerberg & Phillips (1972). Of particular interest are the unsteady eigensolutions that form part of the far-downstream flow. One set of these was studied by Lam & Rott (1960), Ackerberg & Phillips (1972) and Goldstein (1983) and has an exponentially decaying solution downstream (see (8.1) below), with the feature of *decreasing* decay rate with *increasing* order; these eigensolutions are determined primarily by conditions close to the wall. A second set of eigensolutions was constructed by Brown & Stewartson (1973*a*, *b*) and has the feature of *increasing* decay rate with *increasing* order; these eigensolutions are determined from conditions far away from the wall, in the outer reaches of the boundary layer.

Indeed, these seemingly diverse characteristics of the eigensolutions have been the subject of some controversy over the years. However, Goldstein, Sockol & Sanz (1983) include a quite detailed discussion of this dichotomy; briefly, these authors expound the argument that the two sets of eigensolutions are in fact, equivalent, but with the Brown & Stewartson (1973*a*, *b*) expansions being valid at much longer distances $(O(\ln \frac{1}{2}x)^{\frac{1}{2}} \gg 1)$ downstream than the Lam & Rott (1960) eigensolutions (which are valid for $O(x) \gg 1$). Further, Goldstein *et al.* (1983) point out that as the order of the Lam & Rott (1960) eigensolutions increases, the asymptotic behaviour of the (inner) solution is likely to be achieved at progressively larger values of x, since, for $x \gg 1$, the region associated with the eigensolutions moves away from the wall with increasing order. This, in some ways is not inconsistent with the fact that the Brown & Stewartson (1973*a*, *b*) eigensolutions are centred at the outer edge of the steady boundary layer. Goldstein *et al.* (1983) also conclude, using these arguments, that the limit as $x \rightarrow \infty$ and the limit as $n \rightarrow \infty$ (where n is the order of the eigensolution) cannot be interchanged. However, and significantly, Goldstein (1983) went on to illustrate the physical importance of the Lam & Rott (1960) eigensolutions, by showing how these develop, far downstream, into unstable Tollmien–Schlichting waves.

The problem of 'order-one' unsteady, free-stream disturbances (but such that the free stream does not reverse direction) has been considered by a number of authors. Pedley (1972) considered this problem, asymptotically close to and far from the leading edge, whilst Phillips & Ackerberg (1973) presented numerical solutions for locations from the leading edge to far downstream, their method being based on a time-marching scheme. More recently, Duck (1989) presented a new numerical method to tackle this problem, based on a spectral treatment in time, and a spatial finite-difference scheme, which properly takes into account the regions of reversed flow that inevitably occur.

The problem of steady flow along a circular cylinder (in particular far downstream along the axis of the cylinder) is itself interesting, partly because it is so very different in nature from that of planar (i.e. Blasius-type) flow. Early investigations of this problem include the work of Glauert & Lighthill (1955) and Stewartson (1955), whilst Bush (1976) has presented a more modern approach. Notably, in the far downstream limit, the problem becomes double structured, with an inner layer (comparable in thickness with the radius of the body) which is predominantly viscous in nature, and an outer layer (much larger than the radius of the body) which is a region of predominantly uniform flow (see §4 for fuller details).

In this paper we investigate the effect of small-amplitude, time-periodic, free-stream disturbances on the axisymmetric boundary layer on a circular cylinder. Particular emphasis is placed on the eigensolutions relevant to the far-downstream flow, which turn out to be markedly different from the analogous planar eigensolutions of Lam & Rott (1960), and possess some interesting properties. Further, since an additional lengthscale is present in the problem (i.e. the body radius), a second non-dimensional parameter (in addition to the Reynolds number) is present, and we are able to exploit this parameter from an asymptotic point of view.

The layout of the paper is as follows. In §2 the problem is formulated, and in §3 a fully numerical finite-difference scheme for the steady and unsteady problem is described, and results for the wall shears are presented, for axial locations from the leading edge to far downstream. The development of the (inhomogeneous) component of the flow is described in §4. In §5 the presence of eigensolutions far downstream is elucidated, and the eigenproblem is formulated and expanded in the form of an asymptotic series. In §6 numerical solutions of the (leading-order) eigenproblem are described, whilst in §7 the eigenproblem is considered in the asymptotic limits of high and low free-stream oscillation. The conclusions of the paper are presented in §8.

2. Formulation

We introduce a cylindrical polar coordinate system (ar, θ, az), where a is the radius of the body (assumed constant), and the z-axis lies along the axis of the body, with $x = 0$ corresponding to the tip of the body.

Suppose that the fluid is incompressible and of kinematic viscosity ν, and the free-stream velocity is taken to be purely in the z-direction and of the form $W_\infty(1+\delta \cos\omega t^*)$, where W_∞, δ and ω are constants, with $\delta \ll 1$. Note that although in all the ensuing analysis we shall confine our attention exclusively to free-stream velocities of the above form, it is relatively straightforward to extend our ideas to other temporal (periodic) variations.

The velocity field is written as $W_\infty(u,0,w)$, and non-dimensional time as $t=\omega t^*$. Further, it is assumed that u, w, and indeed the entire solution is independent of θ, implying axial symmetry.

In this problem there are two fundamental non-dimensional parameters, namely a Reynolds number based on cylinder radius

$$R=\frac{W_\infty a}{\nu}, \tag{2.1}$$

which will be assumed to be large throughout this paper, together with a frequency parameter

$$\beta=\frac{\nu}{\omega a^2}. \tag{2.2}$$

The usage of the boundary-layer approximation requires that

$$Z=R^{-1}z \tag{2.3}$$

is the key axial lengthscale, and

$$U=Ru \tag{2.4}$$

is the important order-one radial velocity scale. The boundary-layer equations then become (to leading order)

$$\frac{1}{\beta}\frac{\partial w}{\partial t}+w\frac{\partial w}{\partial Z}+U\frac{\partial w}{\partial r}=\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{\beta}\frac{\partial w}{\partial t} \quad (r\rightarrow\infty), \tag{2.5}$$

together with

$$\frac{\partial}{\partial Z}(rw)+\frac{\partial}{\partial r}(rU)=0. \tag{2.6}$$

Since it will be assumed that $\delta \ll 1$, the unsteady component of the flow may be taken to be a small perturbation about the steady solution (a similar treatment has been used in many of the related planar studies cited in the previous section, for example Lam & Rott 1960; Lighthill 1954; Ackerberg & Phillips 1972). Specifically

$$U(r,Z,t)=U_0(r,Z)+\delta\,\mathrm{Re}\,\{\tilde{U}(r,Z)\,\mathrm{e}^{\mathrm{i}t}\}+O(\delta^2), \tag{2.7}$$

$$w(r,Z,t)=w_0(r,Z)+\delta\,\mathrm{Re}\,\{\tilde{w}(r,Z)\,\mathrm{e}^{\mathrm{i}t}\}+O(\delta^2). \tag{2.8}$$

The steady component of the solution is described by

$$w_0\frac{\partial w_0}{\partial Z}+U_0\frac{\partial w_0}{\partial r}=\frac{\partial^2 w_0}{\partial r^2}+\frac{1}{r}\frac{\partial w_0}{\partial r}, \tag{2.9}$$

$$\frac{\partial}{\partial Z}(rw_0)+\frac{\partial}{\partial r}(rU_0)=0, \tag{2.10}$$

with

$$w_0(r=1)=U_0(r=1)=0, \qquad w_0\rightarrow 1 \quad \text{as} \quad r\rightarrow\infty, \tag{2.11}$$

whilst the unsteady perturbation to this flow is given by

$$\frac{\mathrm{i}\tilde{w}}{\beta}+w_0\frac{\partial \tilde{w}}{\partial Z}+\tilde{w}\frac{\partial w_0}{\partial Z}+U_0\frac{\partial \tilde{w}}{\partial r}+\tilde{U}\frac{\partial w_0}{\partial r}=\frac{\partial^2 \tilde{w}}{\partial r^2}+\frac{1}{r}\frac{\partial \tilde{w}}{\partial r}+\frac{\mathrm{i}}{\beta}, \tag{2.12}$$

$$\frac{\partial}{\partial Z}(r\tilde{w})+\frac{\partial}{\partial r}(r\tilde{U})=0, \tag{2.13}$$

subject to

$$\tilde{w}(r=1)=\tilde{U}(r=1)=0, \qquad \tilde{w}\rightarrow 1 \quad \text{as} \quad r\rightarrow\infty. \tag{2.14}$$

To close the problem we further suppose that as $Z\rightarrow 0$, planar conditions prevail, with the boundary-layer thickness becoming negligible in this limit. A similar procedure was followed by Seban & Bond (1951) and was further utilized in a related problem by Duck & Bodonyi (1986). The (steady) system (2.9)–(2.11) then reduces to the Blasius (planar) problem as $Z\rightarrow 0$, with corrections due to curvature effects given by Seban & Bond (1951). As $Z\rightarrow\infty$, the far downstream, double-structured solution of Glauert & Lighthill (1955), Stewartson (1955) and Bush (1976) emerges from this system. The unsteady system (2.12)–(2.14) becomes quasi-steady in form as $Z\rightarrow 0$, with the time-derivative term vanishing in this limit.

In the following section fully numerical solutions to both the steady and unsteady systems are considered, and in the later sections of this paper the far-downstream behaviour of the unsteady component of the flow is investigated in some detail.

3. Numerical solution of the problem

In this section we consider fully numerical solutions to systems (2.9)–(2.11) and (2.12)–(2.14).

Two stream functions are introduced, one for the steady component of the flow, the other for the unsteady component, namely Ψ and $\tilde{\psi}$, respectively given by

$$U_0=-\frac{1}{r}\frac{\partial \Psi}{\partial Z}, \quad w_0=\frac{1}{r}\frac{\partial \Psi}{\partial r}, \tag{3.1}$$

$$\tilde{U}=-\frac{1}{r}\frac{\partial \tilde{\psi}}{\partial Z}, \quad \tilde{w}=\frac{1}{r}\frac{\partial \tilde{\psi}}{\partial r}. \tag{3.2}$$

The problem determining Ψ and $\tilde{\psi}$ is then

$$\Psi_{rrr}-\frac{\Psi_{rr}}{r}+\frac{\Psi_r}{r^2}=\frac{1}{r}\Psi_r\Psi_{rZ}-\Psi_Z\left[\frac{\Psi_{rr}}{r}-\frac{1}{r^2}\Psi_r\right], \tag{3.3}$$

$$\tilde{\psi}_{rrr}-\frac{\tilde{\psi}_{rr}}{r}+\frac{\tilde{\psi}_r}{r^2}=\frac{1}{r}\left[\Psi_r\tilde{\psi}_{rZ}+\tilde{\psi}_r\Psi_{rZ}\right]-\Psi_Z\left[\frac{\tilde{\psi}_{rr}}{r}-\frac{\tilde{\psi}_r}{r^2}\right]-\tilde{\psi}_Z\left[\frac{\Psi_{rr}}{r}-\frac{\Psi_r}{r^2}\right]+\mathrm{i}\frac{\tilde{\psi}_r}{\beta}-\frac{\mathrm{i}r}{\beta}, \tag{3.4}$$

with

$$\Psi=\Psi_r=\tilde{\psi}=\tilde{\psi}_r=0 \quad \text{on} \quad r=1, \tag{3.5}$$

$$\Psi_r,\tilde{\psi}_r\rightarrow r \quad \text{as} \quad r\rightarrow\infty. \tag{3.6}$$

Anticipating a Blasius-type solution as $Z\rightarrow 0$, the problem for $0<Z\leqslant 1$ was cast in terms of

$$\bar{\eta}=(r-1)/Z^{\frac{1}{2}}, \quad \zeta=Z^{\frac{1}{2}}, \tag{3.7}$$

as the independent variables, with the dependent variables taken as F_0 and $\tilde{F}$, where

$$\Psi=\zeta F_0(\bar{\eta},\zeta), \quad \tilde{\psi}=\zeta\tilde{F}(\bar{\eta},\zeta). \tag{3.8}$$

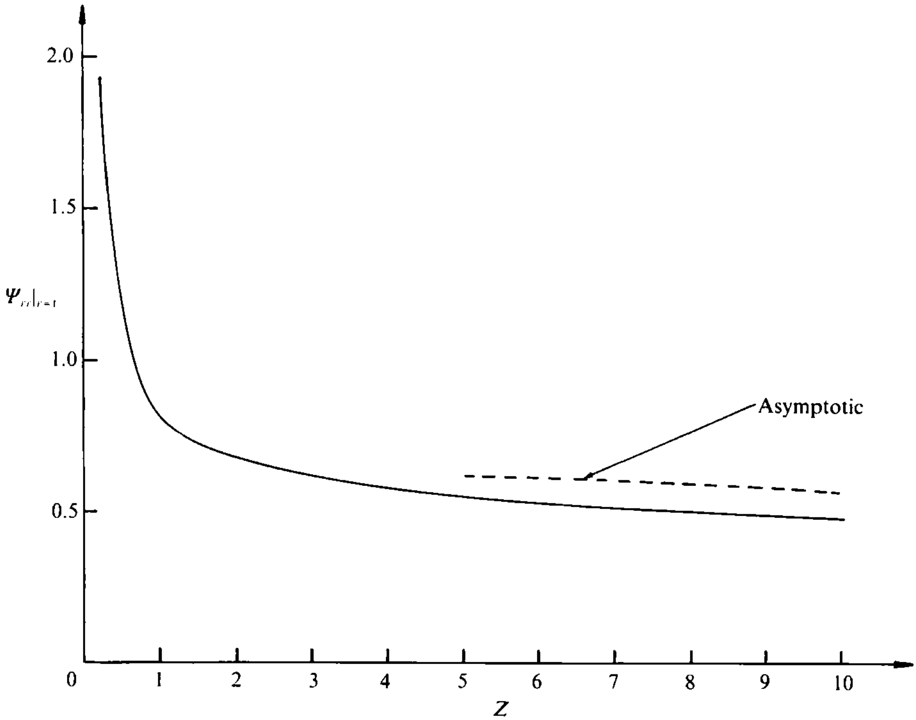

FIGURE 1. Variation of $\Psi_{0rr}|_{r=1}$.

For $Z > 1$, $\Psi(r, Z)$ and $\tilde{\psi}(r, Z)$ were treated as the unknown variables. In both $Z \leqslant 1$ and $Z > 1$ the systems were written as a set of first-order equations in r (or $\bar{\eta}$). Having solved the problem for $Z = 0$, a Crank–Nicolson procedure in Z (or ζ) was adopted. Overall, the numerical differencing scheme was based on that of Keller & Cebeci (1971). At each Z (or ζ) station, first the steady system was computed, with Newton iteration being used to treat the nonlinearity in the problem. Once convergence was achieved, the (linear) unsteady system was then computed in a straightforward manner.

Results for $\Psi_{rr}|_{r=1}$ (essentially the steady component of wall shear) along the cylinder are shown in figure 1. This illustrates the (Blasius-type) singularity as $Z \to 0$, together with a monotonic decline as Z increases.

Figure 2 shows the results for the real and imaginary components of $\tilde{\psi}_{rr}|_{r=1}$ (essentially the unsteady component of wall shear) for $\beta = 0.25$. This shows how the real component exhibits an inverse square singularity as $Z \to 0$ (in line with that of $\Psi_{rr}|_{r=1}$) whilst the imaginary component drops to zero at the leading edge. This occurs because as stated previously, as $Z \to 0$, the system determining $\tilde{F}(\bar{\eta}, \zeta)$ becomes quasi-steady, with the unsteady velocity perturbation moving entirely in phase across the boundary layer. For $Z \geqslant 1$, both the real and imaginary components rapidly approach constant values. This aspect is dealt with in the following section.

Figures 3, 4 and 5 show the corresponding distributions for $\beta = 1$, 2 and 5 respectively, all of which exhibit similar qualitative features to the $\beta = 0.25$ results, although the asymptotic amplitude of $\tilde{\psi}_{rr}|_{r=1}$ is seen to diminish as β increases. In the following section the asymptotic form of the flow structure, far downstream of the leading edge is considered.

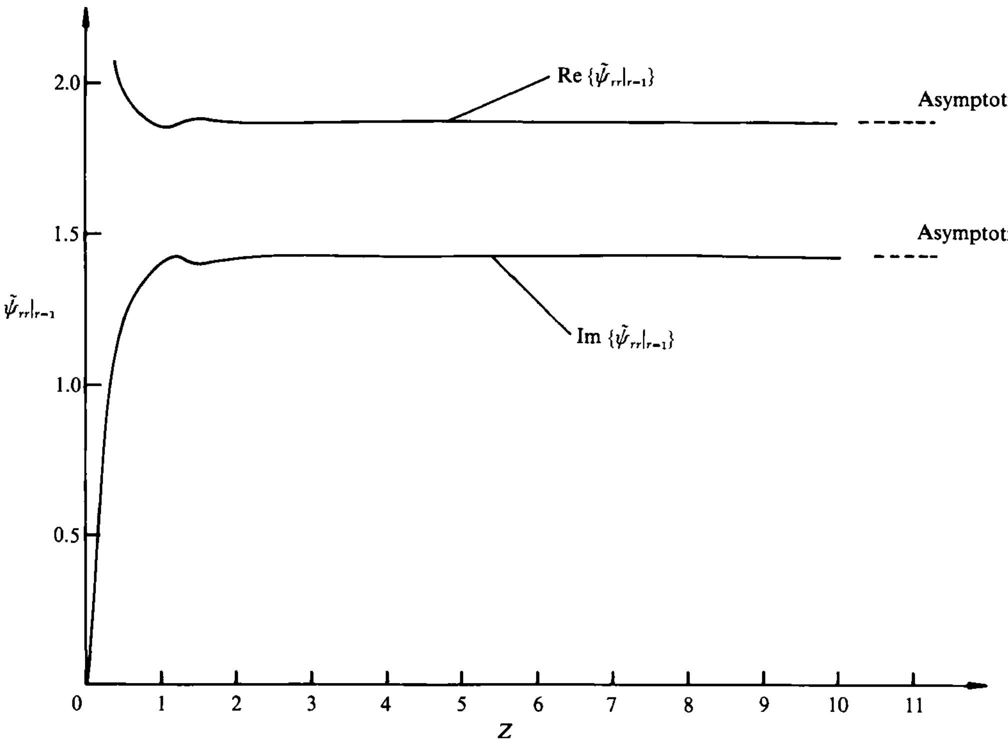

FIGURE 2. Variation of $\tilde{\psi}_{rr}|_{r=1}$, $\beta = \frac{1}{4}$

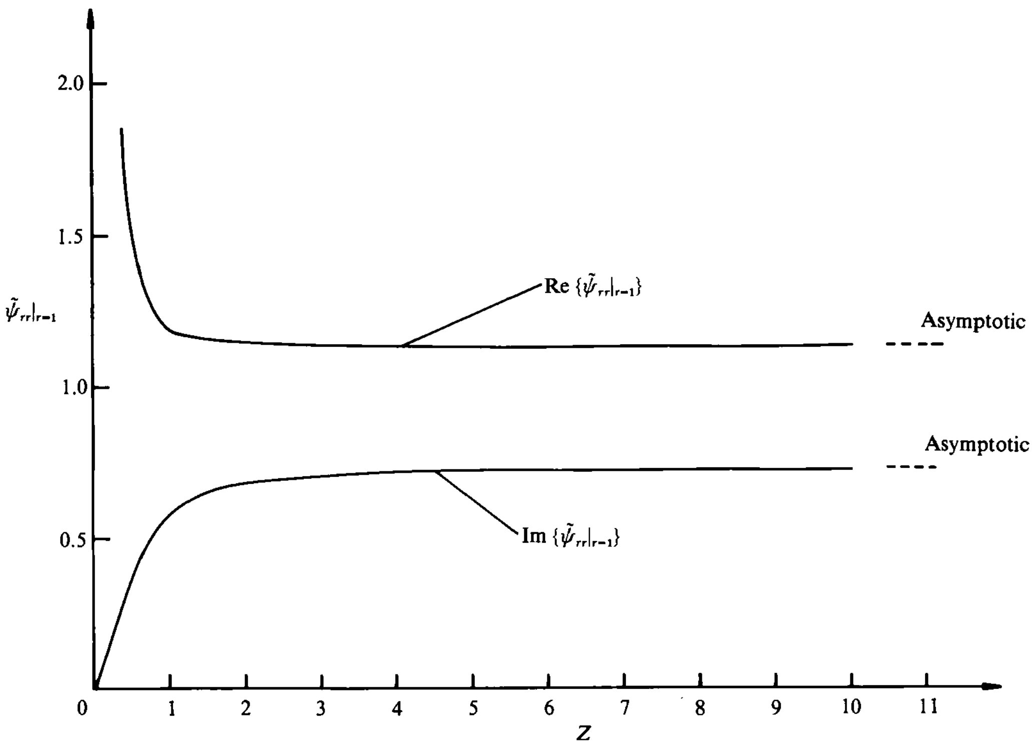

FIGURE 3. Variation of $\tilde{\psi}_{rr}|_{r=1}$, $\beta = 1$.

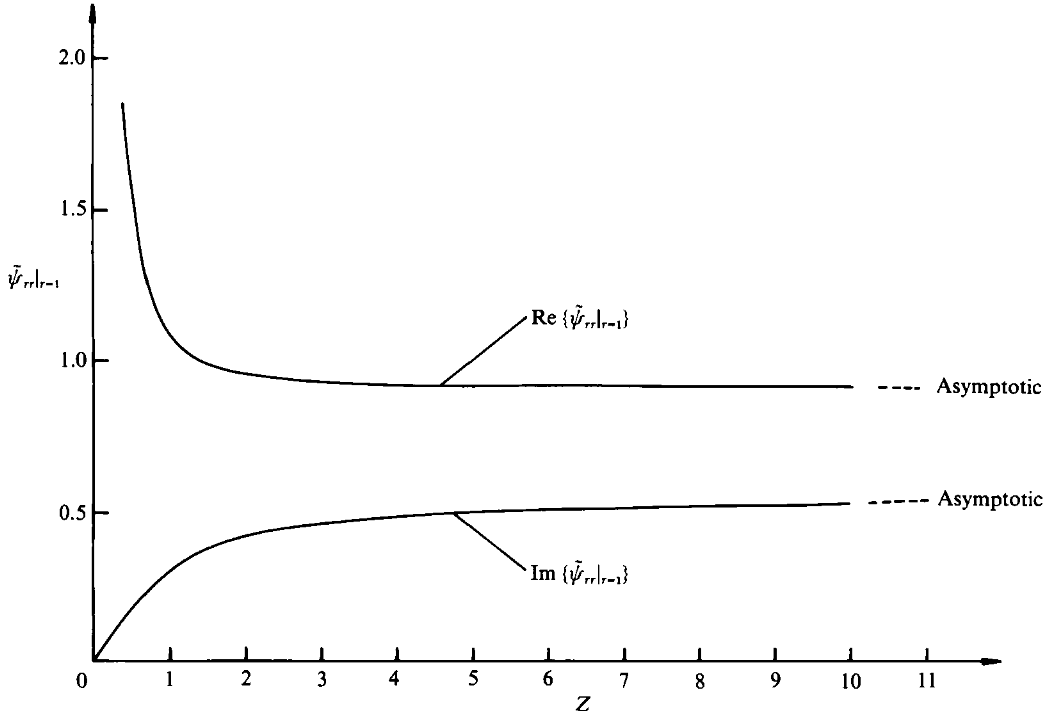

FIGURE 4. Variation of $\tilde{\psi}_{rr}|_{r=1}$, $\beta = 2$.

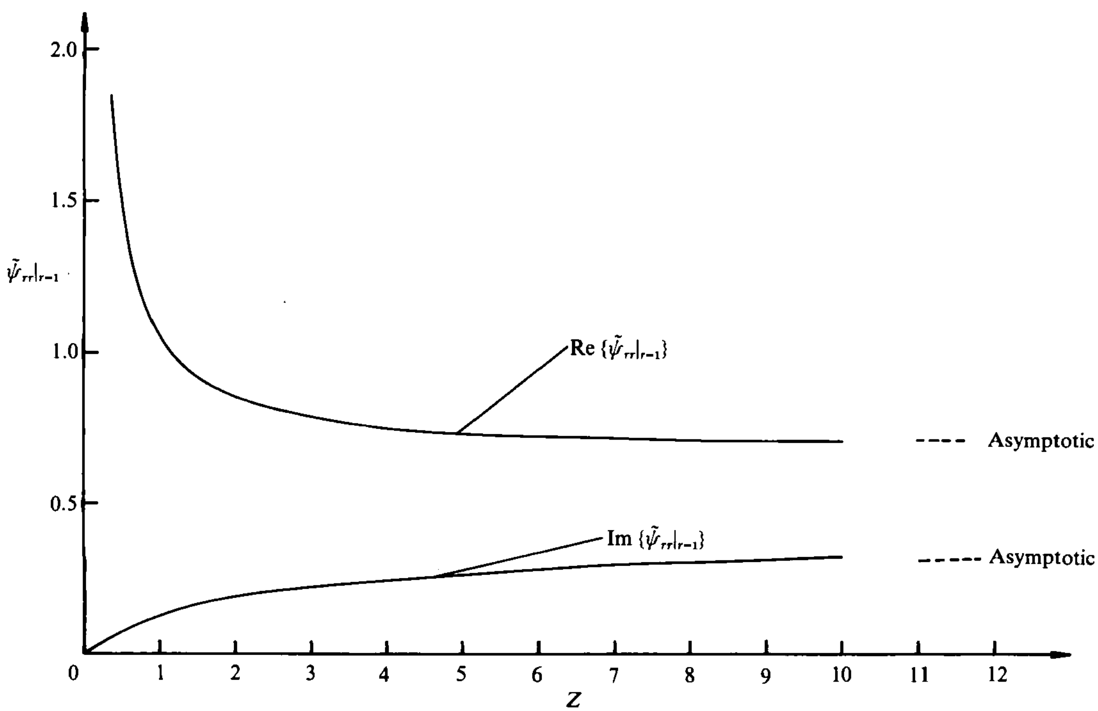

FIGURE 5. Variation of $\tilde{\psi}_{rr}|_{r=1}$, $\beta = 5$.

4. The far downstream development of the flow

In this section we investigate the $Z \gg 1$ solution for the (unsteady) system (2.12)–(2.14). It was shown by Glauert & Lighthill (1955), Stewartson (1955) and Bush (1976) that the steady solution obtained from (2.9)–(2.11) divides into two layers far downstream. Specifically, for $r = O(1)$ it was shown that

$$\Psi = \sum_{n=0}^{\infty} \epsilon^{n+1} \Psi_{0n}(r) + \sum_{n=0}^{\infty} \frac{\epsilon^{n+2}}{Z} \Psi_{1n}(r) + O\left(\frac{1}{Z^2}\right), \tag{4.1}$$

where

$$\epsilon = 2/\log Z, \tag{4.2}$$

and where

$$\left.\begin{aligned} \Psi_{0n}(r) &= K_{0n}\{\tfrac{1}{2}r^2 \log r - \tfrac{1}{4}r^2 + \tfrac{1}{4}\}, \\ K_{00} &= 1, \quad K_{01} = (\tfrac{1}{2}\gamma - \log 2), \\ \gamma &= 0.5772\ldots \quad \text{(Euler's constant)}, \end{aligned}\right\} \tag{4.3}$$

$$\left.\begin{aligned} \Psi_{10} &= K_{10}\{\tfrac{1}{2}r^2 \log r - \tfrac{1}{4}r^2 + \tfrac{1}{4}\}, \\ K_{10} &= \tfrac{7}{4}. \end{aligned}\right\} \tag{4.4}$$

(Note that in Stewartson 1955, the last term in his equation (3.20) should be a logarithmically squared term, and not as shown.) It is also found that

$$\Psi'_{11}(r) = -K_{00}\, r \log r \int_1^r \left\{ \frac{r}{2\Psi'^2_{00}} \int_1^r \frac{\Psi'_{00}}{r} \frac{\mathrm{d}}{\mathrm{d}r}\left[\Psi_{00} \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Psi'_{00}}{r}\right)\right] \mathrm{d}r \right\} \mathrm{d}r$$
$$+ K_{00}\, r \log r \int_1^r \frac{\hat{K}_{11} r}{\Psi'^2_{00}} \mathrm{d}r + K_{11}\, r \log r, \tag{4.5}$$

implying that, for $r \gg 1$,

$$\Psi_{11} \sim \sum_{j=0}^{4} \sum_{n=0}^{\infty} a_{jn} r^j (\log r)^{2-n}. \tag{4.6}$$

Consider now the outer layer, wherein

$$\eta = r/Z^{\frac{1}{2}} = O(1) \tag{4.7}$$

(consistent with (3.3)), and

$$\Psi = Z\{\hat{\Psi}_{00}(\eta) + \epsilon \hat{\Psi}_{01}(\eta) + O(\epsilon^2)\} + \epsilon^2 \hat{\Psi}_{10}(\eta) + O(\epsilon^3). \tag{4.8}$$

It is found $\hat{\Psi}_{00}(\eta) = \frac{1}{2}\eta^2$,

$$\left.\begin{aligned} \hat{\Psi}_{01}(\eta) &= \tfrac{1}{2}\eta^2 \int_\infty^\eta \frac{\mathrm{e}^{-\frac{1}{4}\eta^2}}{\eta} \mathrm{d}\eta + \mathrm{e}^{-\frac{1}{4}\eta^2} - 1, \\ \hat{\Psi}_{10}(\eta) &= \hat{\Psi}_{01}(\eta) - \tfrac{1}{2}. \end{aligned}\right\} \tag{4.9}$$

For comparison with our fully numerical results, asymptotic approximations to the basic flow determined from

$$\Psi_{rr}|_{r=1} \doteqdot \epsilon \Psi_{00rr}(r=1) + \epsilon^2 \Psi_{01rr}(r=1) \doteqdot \epsilon + K_{01}\epsilon^2 \tag{4.10}$$

are shown on figure 1 as a broken line.

Now consider the $Z \gg 1$ solution to the system given by (2.12)–(2.14). This turns out to be quite straightforward. Consider first (and most importantly) the radial scale $r = O(1)$; then owing to the smallness of ϵ, $\tilde{w}$ is expected to develop as

$$\tilde{w}(r, Z) = \tilde{w}_0(r) + O(\epsilon), \tag{4.11}$$

where $\tilde{w}_0$ is to be determined from

$$\frac{\partial^2 \tilde{w}_0}{\partial r^2}+\frac{1}{r}\frac{\partial \tilde{w}_0}{\partial r}-\frac{\mathrm{i}\tilde{w}_0}{\beta}=-\frac{\mathrm{i}}{\beta}, \tag{4.12}$$

the appropriate solution of which is simply

$$\tilde{w}_0(r)=1-\frac{H_0^{(2)}[r(-\mathrm{i}/\beta)^{\frac{1}{2}}]}{H_0^{(2)}[(-\mathrm{i}/\beta)^{\frac{1}{2}}]}, \tag{4.13}$$

i.e. the axisymmetric Stokes shear-wave solution, where $H_0^{(2)}(z)$ denotes the second Hankel function, of order zero and argument z. Note also that as $\beta \rightarrow 0$, the planar Stokes shear solution is retrieved (in accord with the work of Ackerberg & Phillips 1972). In this limit, a thin Stokes layer forms on the surface of the body and consequently curvature effects become less important.

It is a routine matter to continue this solution to higher orders of ϵ; however little additional insight is gleaned from this, and instead we go on to consider (briefly) the outer layer, where $\eta = O(1)$ (see (4.3)).

Writing
$$\tilde{w}=\hat{w}_0(\eta)+\epsilon\hat{w}_1(\eta)+O(\epsilon^2), \tag{4.14}$$
then
$$\hat{w}_0(\eta)=1, \tag{4.15}$$
$$\hat{w}_1(\eta)=\hat{w}_2(\eta)=\ldots=0. \tag{4.16}$$

In fact the correction to $\hat{w}_0(\eta)$ can only be algebraically small in Z^{-1}.

Results obtained using this asymptotic structure (in particular (4.8)) are shown for comparison with the fully numerical results as broken lines on figures 2–5; the agreement is seen to be satisfactory.

However, since the $Z \gg 1$ structure detailed above is obtained without any recourse to upstream conditions, there must be a further element to the downstream flow, not reflected in the above analysis (see also the comments of Ackerberg & Phillips 1972). This arises from eigenfunctions of the system (2.12)–(2.14), which are investigated next, in some detail.

5. The form of the eigensolutions as $Z \rightarrow \infty$

Here the form of (exponentially small) eigensolutions as $Z \rightarrow \infty$ is sought. Specifically, we investigate eigensolutions of (3.4), with the basic flow described by §4.

As a first approximation to the form of these eigensolutions, consider the scale $r = O(1)$, and suppose Ψ in (3.4) is replaced by $\epsilon \Psi_{00}(r)$, and terms $O(\tilde{\psi}/Z)$ and smaller are neglected. This yields

$$\tilde{\psi}_{rrr}-\frac{\tilde{\psi}_{rr}}{r}+\frac{\tilde{\psi}_r}{r^2}-\mathrm{i}\frac{\tilde{\psi}_r}{\beta}-\frac{\epsilon}{r}\Psi_{00r}\tilde{\psi}_{rZ}-\epsilon\tilde{\psi}_Z\left[\frac{\Psi_{00rr}}{r}-\frac{\Psi_{00r}}{r^2}\right]=0. \tag{5.1}$$

Assuming a solution for $\tilde{\psi}$ by separation of variables, namely

$$\tilde{\psi}=f(Z)\,\psi(r), \tag{5.2}$$

then

$$\psi_{rrr}-\frac{\psi_{rr}}{r}+\frac{\psi_r}{r^2}-\mathrm{i}\frac{\psi_r}{\beta}-\epsilon\frac{f_Z}{f}\left\{\frac{1}{r}\Psi_{00r}\psi_r+\psi\left[\frac{\psi_{00rr}}{r}-\frac{\Psi_{00r}}{r^2}\right]\right\}=0. \tag{5.3}$$

A solution of the assumed form is possible only if

$$f_Z+\Lambda/\epsilon f=0. \tag{5.4}$$

where A is a constant. Recalling the definition of ϵ in (4.2), (5.4) integrates to give

$$f(Z) = \exp\{-\tfrac{1}{2}A[Z\log Z - Z]\} = Z^{-\frac{1}{2}AZ}\,\mathrm{e}^{\frac{1}{2}AZ}. \tag{5.5}$$

Here it is required that $\mathrm{Re}(A) > 0$ to ensure decay as $Z \to \infty$, and the arbitrary multiplicative constant in $\psi(r)$ has been included.

However (5.2) and (5.5) are correct only to leading order in ϵ and Z. It turns out the form of $\tilde{\psi}$ required for $r = O(1)$ is

$$\tilde{\psi} = h(Z)f(Z)\,Z^p(\log Z)^q\{\psi_{00}(r) + \epsilon\psi_{01}(r) + O(\epsilon^2) + (1/Z)\,[\epsilon\tilde{\psi}_{10}(r) + O(\epsilon^2)] + O(1/Z^2)\}, \tag{5.6}$$

where $f(Z)$ is given by (5.5) and $h(Z)$ is smaller than any power of $\log Z$; p and q are constants to be determined at some later stage. Further, it is found necessary to expand A itself in terms of ascending powers of ϵ, namely

$$A = A_0 + \epsilon A_1 + \epsilon^2 A_2 + O(\epsilon^3). \tag{5.7}$$

In view of our comments regarding $\mathrm{Re}(A)$, then $\mathrm{Re}(A_0) > 0$. The form of (5.6) and (5.7) is necessitated because of the series development of the basic flow in powers of ϵ and $1/Z$, and is found to be essential for solubility at higher orders of the solution. (Indeed Goldstein 1983 pointed out the omission of algebraic terms in the streamwise development of the planar eigensolutions in the work of Ackerberg & Phillips 1972, which contained only the exponential development of the flow.)

Substitution of (5.6) and the results of §4 into (3.4), and taking terms $O(h(Z)\,Z^{-\frac{1}{2}A_0Z}\,\mathrm{e}^{\frac{1}{2}A_0Z}\,Z^p(\log Z)^q)$ yields the following equation for ψ_{00}:

$$L\{\Psi_{00}\} = 0, \tag{5.8}$$

where

$$L\{\psi_{00}\} \equiv \psi_{00}''' - \frac{\psi_{00}''}{r} + \psi_{00}'\left[\frac{1}{r^2} - \frac{\mathrm{i}}{\beta} + \frac{A_0}{r}\Psi_{00}'\right] - A_0\,\psi_{00}\left[\frac{\Psi_{00}''}{r} - \frac{\Psi_{00}'}{r^2}\right]. \tag{5.9}$$

Recalling the form of Ψ_{00}, given in (4.3), then

$$L\{\psi_{00}\} = \psi_{00}''' - \frac{\psi_{00}''}{r} + \psi_{00}'\left[\frac{1}{r^2} - \frac{\mathrm{i}}{\beta} + A_0\log r\right] - A_0\frac{\psi_{00}}{r} = 0. \tag{5.10}$$

The boundary conditions to be applied to this system are those of no-slip and impermeability on $r = 1$, i.e.

$$\psi_{00}(r=1) = \psi_{00}'(r=1) = 0, \tag{5.11}$$

whilst as $r \to \infty$, ψ_{00} should *not* be exponentially large. To be more precise on this last point, the three linearly independent solutions to (5.10) in this limit take the form

$$\psi_{00}^A \sim A_{00}^A\{\log r - (\mathrm{i}/\beta A_0)\}, \tag{5.12}$$

$$\psi_{00}^B \sim \frac{A_{00}^B}{(\log r)^{\frac{3}{4}}}\exp\left[\mathrm{i}\int_1^r (A_0\log r)^{\frac{1}{2}}\,\mathrm{d}r\right], \tag{5.13}$$

$$\psi_{00}^C \sim \frac{A_{00}^C}{(\log r)^{\frac{3}{4}}}\exp\left[-\mathrm{i}\int_1^r (A_0\log r)^{\frac{1}{2}}\,\mathrm{d}r\right]. \tag{5.14}$$

Clearly one of (5.13) or (5.14) is inadmissible (if A_0 is complex) due to the $r \gg 1$ condition, and so

$$\psi_{00} = A_{00}\left[\log r - \frac{\mathrm{i}}{\beta A_0} + \frac{2}{r^2 A_0\log r} + O\left(\frac{1}{r^2(\log r)^2}\right)\right] \tag{5.15}$$

in this limit, where A_{00} is an arbitrary constant (amplitude). The system (5.10), (5.11) and (5.15) represents an eigenvalue problem for A_0. However we defer discussion of this problem until the following section (where a detailed investigation is carried out of this aspect). Instead, let us consider higher-order terms in the expansions (5.6) and (5.7). Taking terms

$$O(h(Z)\,Z^{-\frac{1}{8}A_0 Z}\,e^{\frac{1}{8}A_0 Z}\,Z^p\,(\log Z)^{q-1})$$

in (3.4) yields

$$L\{\psi_{01}\} = A_1\left\{-\frac{\psi'_{00}\Psi'_{00}}{r}+\psi_{00}\left(\frac{\Psi''_{00}}{r}-\frac{\psi'_{00}}{r^2}\right)\right\}+A_0\left\{-\frac{\psi'_{00}\Psi'_{01}}{r}+\psi_{00}\left(\frac{\Psi''_{01}}{r}-\frac{\Psi'_{01}}{r^2}\right)\right\}. \quad (5.16)$$

However, on account of (4.3) this equation may be written as

$$L\{\psi_{01}\} = (A_1+K_{01}A_0)\left\{-\frac{\psi'_{00}\Psi'_{00}}{r}+\psi_{00}\left(\frac{\Psi''_{00}}{r}-\frac{\psi'_{00}}{r^2}\right)\right\}. \quad (5.17)$$

The boundary conditions for this system are essentially the same as those for ψ_{00}; these can only be satisfied if

$$A_1 = -K_{01}A_0, \quad (5.18)$$

implying that

$$\psi_{01}(r) = A_{01}\,\psi_{00}(r), \quad (5.19)$$

where A_{01} is a constant (amplitude). It is straightforward to determine higher-order terms in the A-expansion, in a similar fashion. For example

$$A_2 = A_0K_{01}^2 - A_0K_{02} - \tfrac{1}{2}K_{01}A_0 \quad (5.20)$$

and hence

$$\psi_{02} = A_{02}\,\psi_{00}(r). \quad (5.21)$$

Indeed, the following general result is applicable:

$$\psi_{0n} = A_{0n}\,\psi_{00}(r). \quad (5.22)$$

To progress further, in particular to determine terms that are $O(Z^{-1})$ smaller than those considered already, let us investigate terms

$$O(h(Z)\,Z^{-\frac{1}{8}A_0 Z}\,e^{\frac{1}{8}A_0 Z}\,Z^{p-1}\,(\log Z)^{q-1})$$

in the governing equation. This yields the following equations for ψ_{10}:

$$L\{\psi_{10}\} = pR_1 - A_0R_2, \quad (5.23)$$

where

$$\left.\begin{aligned} R_1 &= \frac{\psi'_{00}\Psi'_{00}}{r}+\psi_{00}\frac{\Psi'_{00}}{r^2}-\frac{\psi_{00}\Psi''_{00}}{r}, \\ R_2 &= \frac{\psi'_{00}\Psi'_{10}}{r}+\frac{\psi_{00}\Psi'_{10}}{r^2}-\frac{\psi_0\Psi''_{10}}{r}. \end{aligned}\right\} \quad (5.24)$$

In view of (4.4)

$$\Psi_{10} = K_{10}\Psi_{00}, \quad (5.25)$$

and so

$$R_2 = K_{10}R_1. \quad (5.26)$$

Repeating the arguments used previously to determine A_1 and A_2, then

$$p = A_0K_{10} = \tfrac{7}{4}A_0. \quad (5.27)$$

Finally for this section, let us consider briefly the outer solution, applicable to the $\eta = O(1)$ scale. In view of the $r = O(1)$ solution, in particular its $O(\log r)$ behaviour as $r\to\infty$, together with (5.6), then for $\eta = O(1)$ the solution is expected to develop in the following form:

$$\tilde{\psi}(\eta, Z) = g(Z)\,h(Z)\,Z^p\,(\log Z)^q\,\{\tilde{\psi}_0(\eta,\epsilon)+\epsilon/Z\,\tilde{\psi}_1(\eta,\epsilon)+O(Z^{-2})\}, \quad (5.28)$$

where
$$g(Z) = f(Z)/\epsilon. \tag{5.29}$$
It is then possible to obtain an exact solution for $\tilde{\psi}_0$ which matches on to the $r = O(1)$ solution. This is given by
$$\tilde{\psi}_0 = \tilde{A}_0\left[A_0 \frac{\hat{\Psi}_{0n}}{\eta} - \frac{\mathrm{i}}{\beta}\right], \tag{5.30}$$
where $\tilde{A}_0$ is a constant, and
$$\hat{\Psi}_0 = \sum_{n=0}^{\infty} \epsilon^n \hat{\Psi}_{0n}(\eta). \tag{5.31}$$
If we now expand
$$\tilde{\psi}_0 = \sum_{n=0}^{\infty} \epsilon^n \tilde{\psi}_{0n}(\eta), \tag{5.32}$$
then it is straightforward to show that
$$\tilde{\psi}_{00} = A_{00} \tag{5.33}$$
(which matches on to (5.15)), and
$$\tilde{\psi}_{01} = A_{00} \frac{\hat{\Psi}_{01\eta}}{\eta} + \bar{A}_{01}, \tag{5.34}$$
where $\bar{A}_{01}$ is an arbitrary constant. Other terms may be obtained similarly.

In the following section we go on to consider numerical solutions to (5.10). The value of q is determined in the Appendix.

6. Numerical solutions of the eigenvalue problem (5.10) and (5.11)

The problem was tackled using three separate numerical techniques. The first comprised a fourth-order Runge–Kutta technique, shooting inwards from $r = r_\infty$ (chosen to be suitably large). In particular, the technique involved (i) imposing a solution of the form (5.12) at r_∞, generating values of ψ_{00}^A $(r = 1)$ and $\psi_{00}^{A\prime}$ $(r = 1)$ and then (ii) imposing a solution of the form (5.13) at r_∞ (or (5.14) depending on the sign of $\mathrm{Re}\{\mathrm{i}A_0^{\frac{1}{3}}\}$), generating values of ψ_{00}^B $(r = 1)$ and $\psi_{00}^{B\prime}$ $(r = 1)$ (or ψ_{00}^C $(r = 1)$ and $\psi_{00}^{C\prime}$ $(r = 1)$). The value of A_0 was then chosen by Newton iteration by imposing (5.11), by forcing the determinant
$$\begin{vmatrix} \psi_{00}^A(r=1) & \psi_{00}^B(r=1) \\ \psi_{00}^{A\prime}(r=1) & \psi_{00}^{B\prime}(r=1) \end{vmatrix} \tag{6.1}$$
or
$$\begin{vmatrix} \psi_{00}^A(r=1) & \psi_{00}^C(r=1) \\ \psi_{00}^{A\prime}(r=1) & \psi_{00}^{C\prime}(r=1) \end{vmatrix} \tag{6.2}$$
to zero.

The second numerical scheme employed involved using a second-order finite-difference approximation to (5.10), constructing a quadra-diagonal system (corresponding to the approximation to (5.10), together with the boundary conditions (5.11), (5.15)), and the determinant of this system was forced to zero by adjusting A_0 by Newton iteration.

The third numerical scheme used was a direct (global) finite-difference approach; using the same finite-difference scheme as our second scheme, the system was instead written in the form
$$\boldsymbol{A} - A_0 \boldsymbol{B} = 0. \tag{6.3}$$
where $\boldsymbol{A}$ and $\boldsymbol{B}$ are both square matrices. The A_0 were determined by using NAG routine F02GJF, suitable for solving generalized eigenvalue problems of this kind.

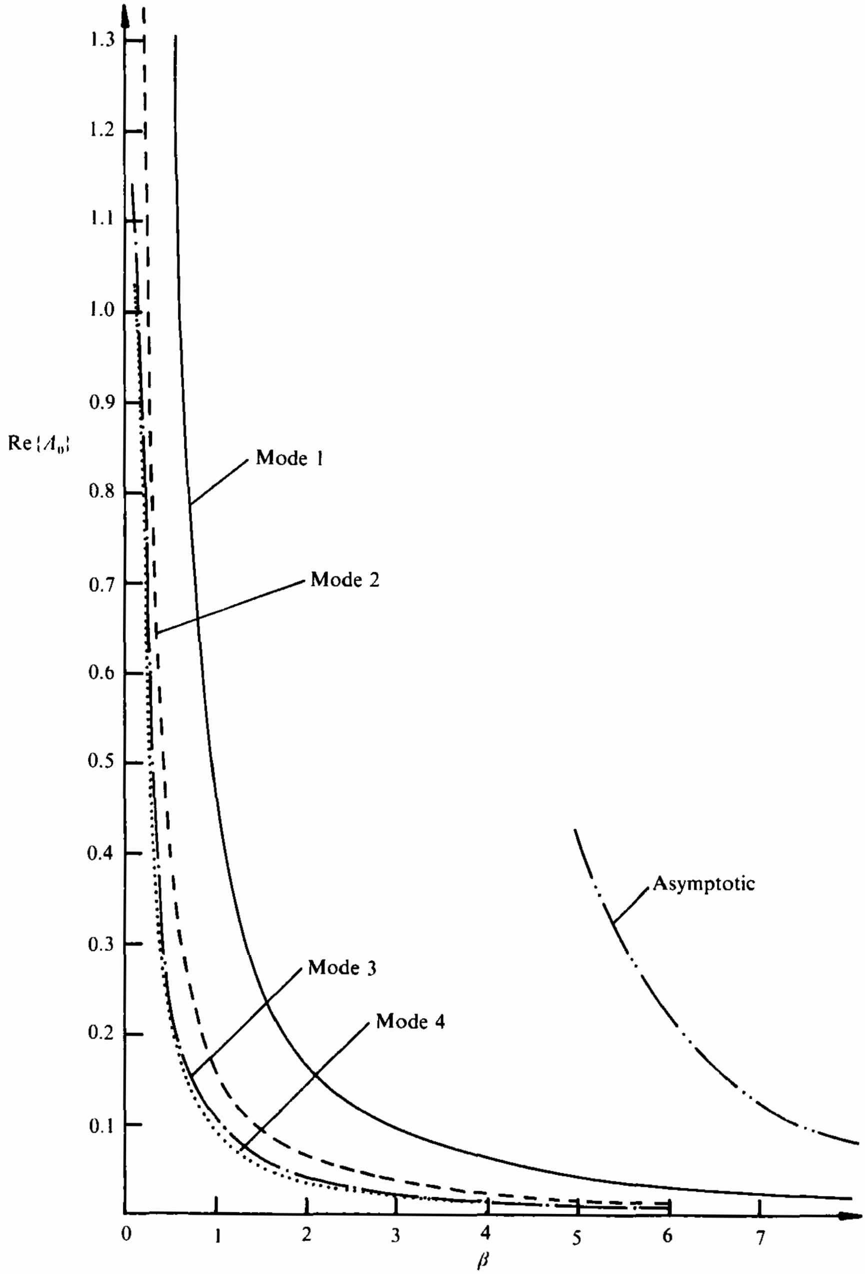

FIGURE 6. Variation of Re{Λ_0} with β.

This scheme has two distinct advantages: (i) not requiring iteration and (ii) generating multiple values (if present) of Λ_0 simultaneously; however, it can require substantial computer storage.

Results from all three schemes were found to agree: in practice the procedure was usually to obtain estimates to the values of Λ_0 using the third scheme. If these were then deemed to be of insufficient accuracy, enhanced solutions (obtained on a finer and/or more extensive grid) were obtained using the second scheme (i.e. the local finite-difference scheme).

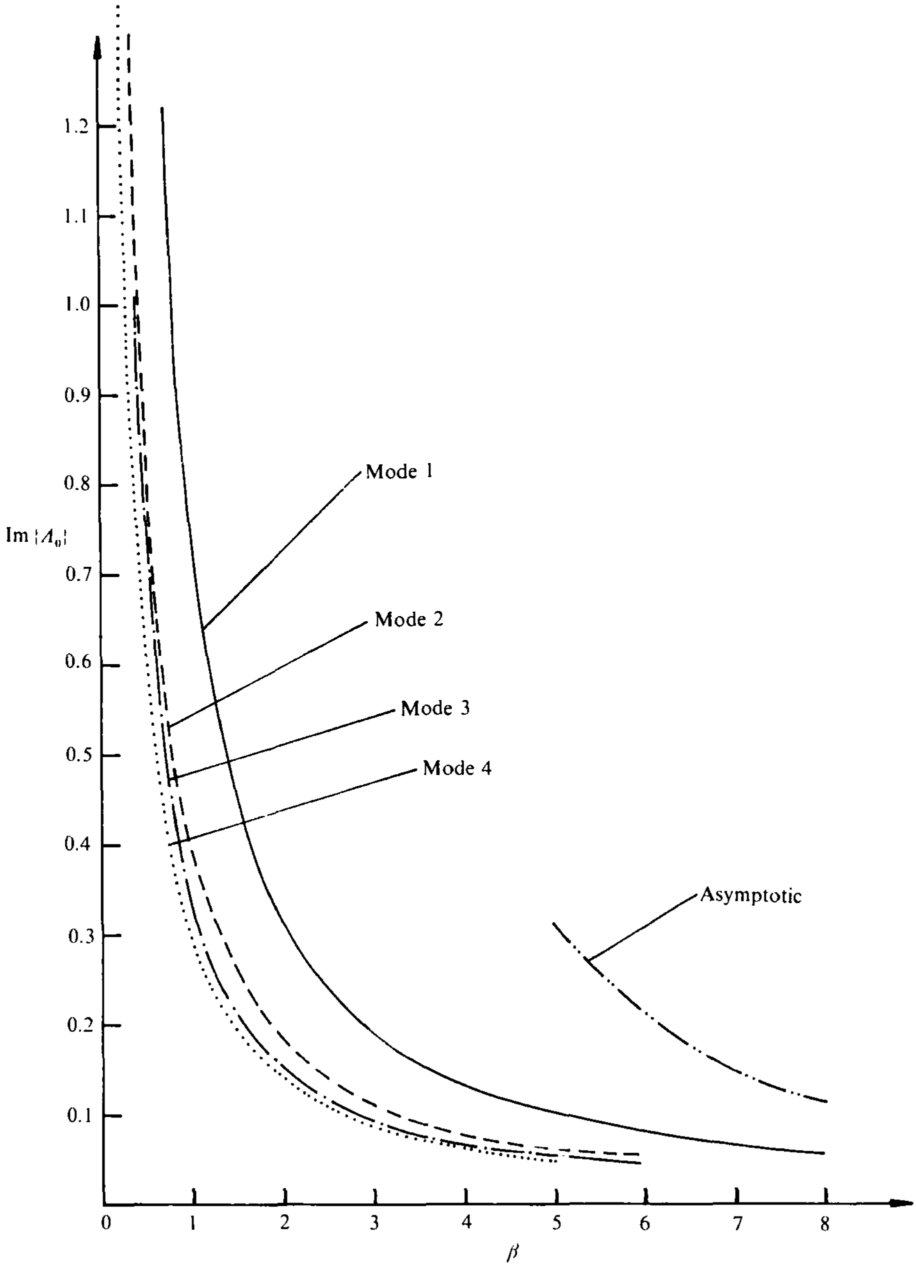

FIGURE 7. Variation of Im $\{A_0\}$ with β.

Results were obtained for a range of β. It was found that at all the values of β investigated there are many (probably an infinite number) values of A_0. Further all three methods did yield a large number of spurious modes. However, these were usually readily identifiable, being strongly dependent upon grid size and range, whilst genuine modes were comparatively grid insensitive.

Results for Re $\{A_0\}$ are shown in figure 6 and for Im $\{A_0\}$ in figure 7. Just the first four modes are shown in each case – higher modes become extremely difficult to compute (and, indeed distinguish from each other and also the previously described spurious modes), particularly in the limits of $\beta \to \infty$ and $\beta \to 0$. However the trends are clear, namely that $|\Lambda_0| \to \infty$ as $\beta \to 0$ and $|\Lambda_0| \to 0$ as $\beta \to \infty$, for all modes. In the following section we investigate these two limits asymptotically.

7. Asymptotic solutions of the eigenvalue problem (5.10)

In this section the limits $\beta \to \infty$ and $\beta \to 0$ in equation (5.10) are considered, for which some analytic progress is possible.

7.1. *The limit* $\beta \to \infty$

Physically, this corresponds to a low-frequency limit to the problem. The numerical results presented in the previous section indicate that (all the) $\Lambda_0 \to 0$ and $\beta \to \infty$. Consequently if $r = O(1)$, then to leading order (assuming $\Lambda_0 = o(1)$)

$$\psi_{00}''' - \frac{\psi_{00}''}{r} + \frac{\psi_{00}'}{r^2} = 0,$$

with

$$\psi_{00}(1) = \psi_{00}'(1) = 0. \tag{7.1}$$

The solution to this system is then

$$\psi_{00} = B_0\{r^2 \log r - \tfrac{1}{2}r^2 + \tfrac{1}{2}\}, \tag{7.2}$$

where B_0 is some arbitrary constant. This solution must ultimately cease to be a valid approximation to (5.10) as $r \to \infty$, specifically when $r = O(\Lambda_0^{-\frac{1}{2}})$. Considering the particular development of Λ_0 as $\beta \to \infty$, this is found to take on the following form, in order to obtain a consistent and meaningful asymptotic solution:

$$\Lambda_0 = \tilde{\gamma}(\beta)\,[\lambda_0 + \hat{\epsilon}\lambda_1 + O(\hat{\epsilon}^2)], \tag{7.3}$$

where

$$\hat{\epsilon} = -2/\log \tilde{\gamma}, \tag{7.4}$$

and $\tilde{\gamma}(\beta)$ must be determined from

$$\tilde{\gamma} \log (\tilde{\gamma}^{-\frac{1}{2}}) = \beta^{-1}. \tag{7.5}$$

This is a transcendental equation for the small parameter $\tilde{\gamma}$ (see Duck 1984; Duck & Hall 1989 for similar examples). In order to obtain a meaningful balance of terms when $r = O(\Lambda_0^{-\frac{1}{2}})$, it is necessary that

$$\lambda_0 = \mathrm{i} \tag{7.6}$$

(the leading term in the expansion for Λ_0).

In view of these comments, and those made above regarding the scale of r for which (7.2) ceases to be a valid approximation to (5.10), we define the outer lengthscale

$$\rho = \tilde{\gamma}^{\frac{1}{2}} r = O(1), \tag{7.7}$$

where the following problem must be considered:

$$\chi_{\rho\rho\rho} - \frac{1}{\rho}\chi_{\rho\rho} + \chi_\rho\left[\frac{1}{\rho^2} + \mathrm{i}\log\rho + \lambda_1\right] - \frac{\mathrm{i}\chi}{\rho} = 0, \tag{7.8}$$

with

$$\left.\begin{aligned} &\chi \sim c_1[\log\rho - \mathrm{i}\,\lambda_1] \quad \text{as} \quad \rho \to \infty, \\ &\chi = O(\rho^2) \quad \text{as} \quad \rho \to 0, \end{aligned}\right\} \tag{7.9}$$

where c_1 is an arbitrary constant, and χ is related to ψ_{00} by

$$\chi = \frac{\tilde{\gamma}}{\log \tilde{\gamma}^{-\frac{1}{2}}}\psi_{00}. \tag{7.10}$$

Mode	λ_1	Asymptotic λ_1
I	$0.785+0.160\mathrm{i}$	
II	$0.785-1.282\mathrm{i}$	$\frac{1}{4}\pi-1.266\mathrm{i}$
III	$0.785-1.934\mathrm{i}$	$\frac{1}{4}\pi-1.959\mathrm{i}$
IV	$0.785-2.340\mathrm{i}$	$\frac{1}{4}\pi-2.364\mathrm{i}$
V	$0.785-2.631\mathrm{i}$	$\frac{1}{4}\pi-2.652\mathrm{i}$
VI	$0.785-2.858\mathrm{i}$	$\frac{1}{4}\pi-2.875\mathrm{i}$
VII	$0.785-3.044\mathrm{i}$	$\frac{1}{4}\pi-3.057\mathrm{i}$
VIII	$0.785-3.200\mathrm{i}$	$\frac{1}{4}\pi-3.211\mathrm{i}$

TABLE 1. Values of λ_1

The system (7.7) and (7.8) represents a well-posed eigenvalue problem for the λ_1 which was solved using the three numerical techniques described in the previous section (indeed (7.8) is very similar to (5.10), and is of about the same computational complexity, save for the absence of any physical parameters).

Values for the first few λ_1 are tabulated in table 1 (accuracy to at least the number of digits shown). It appears that all the λ_1 possessed the *same* real value (and hence decay rate) to within the accuracy of the computation. The evidence was that a large (probably infinite) number of these modes exist; these higher modes were difficult to compute accurately, requiring small grid sizes and extensive grid domains. Further, with increasing order, the imaginary part of the λ_1 became progressively more negative, although the difference between modes did diminish. Indeed, these trends can be confirmed, asymptotically, by carrying out a $|\lambda_1| \gg 1$ analysis on (7.8) and (7.9). In this limit, a WKB solution to (7.8) exists of the form

$$\chi = \frac{B_1 \rho^{\frac{1}{2}}}{[\mathrm{i}\log\rho+\lambda_1]^{\frac{3}{4}}}\exp\left[\int_{\rho_0}^{\rho} \mathrm{i}\,(\mathrm{i}\log\rho+\lambda_1)^{\frac{1}{2}}\,\mathrm{d}\rho\right] + c_1[\log\rho+\mathrm{i}\lambda_1] \tag{7.11}$$

for $\mathrm{Re}\{\log\rho-\mathrm{i}\lambda_1\} > 0$ (and the path of integration lies within this region), where

$$\rho_0 = \mathrm{e}^{\mathrm{i}\lambda_1} \tag{7.12}$$

(and we expect $c_1 = o(B_1)$), whilst

$$\chi = c_1[\log\rho-\mathrm{i}\lambda_1] + \frac{\rho^{\frac{1}{2}}}{[\mathrm{i}\log\rho+\lambda_1]^{\frac{3}{4}}} \times\left\{A_1\exp\left[\int_{\rho_0}^{\rho}\mathrm{i}(\mathrm{i}\log\rho+\lambda_1)^{\frac{1}{2}}\,\mathrm{d}\rho\right] + A_2\exp\left[-\int_{\rho_0}^{\rho}\mathrm{i}[\mathrm{i}\log\rho+\lambda_1]^{\frac{1}{2}}\,\mathrm{d}\rho\right]\right\} \tag{7.13}$$

for $\mathrm{Re}\{\log\rho-\mathrm{i}\lambda_1\} < 0$ (and the path of integration lies within this region).

A routine treatment of the transition layer about $\rho = \rho_0$ reveals

$$A_2 = \mathrm{i}A_1. \tag{7.14}$$

To proceed further, consider an inner layer wherein

$$\rho_1 = \lambda_1^{\frac{1}{2}}\rho = O(1), \tag{7.15}$$

with χ satisfying the following equation to leading order:

$$\chi_{\rho_1\rho_1\rho_1} - \frac{1}{\rho_1}\chi_{\rho_1\rho_1} + \chi_{\rho_1}\left(\frac{1}{\rho_1^2}+1\right) = 0, \tag{7.16}$$

the solution of which is

$$\chi_{\rho_1} = \tfrac{1}{2}B_0\rho_1 J_0(\rho_1) \tag{7.17}$$

(the second solution of this equation involving $Y_0(\rho_1)$ is neglected on account of (7.9)). Taking the limit of (7.17) as $\rho_1 \to \infty$ gives

$$\chi_{\rho_1} \sim B_0\left(\frac{\rho_1}{2\pi}\right)^{\frac{1}{2}} \cos(\rho_1 - \tfrac{1}{4}\pi). \tag{7.18}$$

The limit of (7.13) as $\rho \to 0$ is

$$\chi \to c_1[\log\rho - \mathrm{i}\lambda_1] + \frac{A_1\rho^{\frac{1}{2}}\mathrm{e}^{I_1}}{[\mathrm{i}\log\rho+\lambda_1]^{\frac{3}{4}}}\{\mathrm{e}^{-\mathrm{i}\lambda_1^{\frac{1}{2}}\rho} + \mathrm{i}\,\mathrm{e}^{-2I+\mathrm{i}\lambda_1^{\frac{1}{2}}\rho}\}, \tag{7.19}$$

where

$$I_1 = \int_{\rho_0}^{0} \mathrm{i}[\mathrm{i}\log\rho + \lambda_1]^{\frac{1}{2}}\,\mathrm{d}\rho. \tag{7.20}$$

with the integration path lying within $\mathrm{Re}\{\log\rho - \mathrm{i}\lambda_1\} < 0$. If (7.19) is to match with (7.18) then

$$\mathrm{e}^{2I} = -1, \tag{7.21}$$

which leads to

$$\lambda_1 = \tfrac{1}{4}\pi - \mathrm{i}\log[2\pi^{\frac{1}{2}}n], \tag{7.22}$$

where n is a (large) positive integer. For consistency, we also require

$$B_0 = 2\mathrm{i}\frac{(2\pi)^{\frac{1}{2}}}{\lambda_1^{\frac{1}{4}}}A_0. \tag{7.23}$$

The formula represented by (7.22) was used to obtain asymptotic estimates to the results shown in table 1. Mode II corresponds to $n = 1$, mode III corresponds to $n = 2$ and so on; it is seen that the agreement between the computed asymptotic results is most satisfactory ($\frac{1}{4}\pi = 0.785\ldots$). It is quite clear that this asymptotic form will fail when $n = O(\tilde{\gamma}^{-1})$.

The leading-order terms, namely $\mathrm{Re}(A_0) \doteqdot \frac{1}{4}\pi\tilde{\gamma}\tilde{\epsilon}$ and $\mathrm{Im}(A_0) \doteqdot \tilde{\gamma}$ are shown on figures 6 and 7 respectively, for comparison with the numerical solutions obtained from the full equation (5.10). The results are not contradictory, given the 'largeness' of the small parameter $\hat{\epsilon}$. Indeed, computations for A_0 from (5.10) at larger values of β did become exceedingly difficult, owing to the large lengthscale ($O(\tilde{\gamma}^{-1})$), together with mode 'jumping' caused by the close proximity of modes, which made the use of grid refinement with the local method impractical.

7.2. *The limit $\beta \to 0$*

This corresponds to the high-frequency limit of the problem. According to the numerical results presented in §6, $|A_0|$ increases as $\beta \to 0$. This limit is now investigated.

It is possible to write a WKB-type approximate solution to (5.10) (assuming $|A_0|$ and β^{-1} are both large) as

$$\begin{aligned}\psi_{00}(r) = {} & B_1\left[\log r - \frac{\mathrm{i}}{A_0\beta}\right] + \left[A_0\log r - \frac{\mathrm{i}}{\beta}\right]^{-\frac{5}{4}} \\ & \times r^{\frac{1}{2}}\left\{B_2\exp\left[\mathrm{i}\int_{r_0}^{r}\left[A_0\log r - \frac{\mathrm{i}}{\beta}\right]^{\frac{1}{2}}\mathrm{d}r\right] + B_3\exp\left[-\mathrm{i}\int_{r_0}^{r}\left[A_0\log r - \frac{\mathrm{i}}{\beta}\right]^{\frac{1}{2}}\mathrm{d}r\right]\right\},\end{aligned} \tag{7.24}$$

(for $\mathrm{Re}\{A_0 \log r - (\mathrm{i}/\beta)\} < 0$, and the path of integration lies within this region), where

$$B_3 = \frac{B_2 \mathrm{e}^{2\mathrm{i}I}\{\mathrm{i}(-(\mathrm{i}/\beta))^{\frac{3}{2}} - A_0\}}{\mathrm{i}(-(\mathrm{i}/\beta))^{\frac{3}{2}} + A_0}, \tag{7.25}$$

$$B_1 = -A_0(-(\mathrm{i}/\beta))^{-\frac{1}{4}}[B_2 \mathrm{e}^{\mathrm{i}I} + B_3 \mathrm{e}^{-\mathrm{i}I}], \tag{7.26}$$

and
$$r_0 = \exp[\mathrm{i}/(\beta A_0)] \tag{7.27}$$

is the turning point, and

$$I = -\int_1^{r_0} \left[A_0 \log r - \frac{\mathrm{i}}{\beta}\right]^{\frac{1}{2}} \mathrm{d}r. \tag{7.28}$$

Equations (7.25) and (7.26) are obtained by imposing boundary conditions on $r = 1$, and the integration path lies within $\mathrm{Re}\{A_0 \log r - (\mathrm{i}/\beta) < 0$.

For $\mathrm{Re}\{A_0 \log r - (\mathrm{i}/\beta) > 0$ the WKB-type approximate solutions can be written:

$$\psi_{00}(r) = B_1\left[\log r - \frac{\mathrm{i}}{A_0 \beta}\right] + \frac{B_4 r^{\frac{1}{2}}}{[(\mathrm{i}/\beta) - A_0 \log r]^{\frac{5}{4}}} \exp\left[\int_{r_0}^{r} \left(\frac{\mathrm{i}}{\beta} - A_0 \log r\right)^{\frac{1}{2}} \mathrm{d}r\right], \tag{7.29}$$

where it has been assumed that $\mathrm{Re}\{(-A_0)^{\frac{1}{2}}\} < 0$ (otherwise we require the negative root inside the integral), B_1 is given by (7.26), and the integration path lies within $\mathrm{Re}\{A_0 \log r - (\mathrm{i}/\beta)\} > 0$. In order that (7.29) matches to (7.24) across the transition layer of thickness $O(A_0^{-\frac{1}{3}})$, (routine) treatment (see also the analysis for $A_0 = O(\beta^{-\frac{3}{5}})$ below) of the latter yields

$$B_1 = -\mathrm{i}B_2, \tag{7.30}$$

and the following dispersion relationship for A_0 results:

$$(-(\mathrm{i}/\beta))^{\frac{3}{2}} - \mathrm{i}A_0 = \mathrm{e}^{2\mathrm{i}I}\{\mathrm{i}(-(\mathrm{i}/\beta))^{\frac{3}{2}} - A_0\}. \tag{7.31}$$

It turns out that there are two distinct families of solution as $\beta \to 0$. The first family corresponds to $A_0 = O(\beta^{-1})$. More specifically

$$A_0 = \beta^{-1}[\hat{A}_0 + \beta^{\frac{1}{3}}\hat{A}_1 + \ldots], \tag{7.32}$$

where $\hat{A}_0$, $\hat{A}_1$ are generally $O(1)$ quantities. This implies that $r_0 - 1 = O(1)$. Consequently to leading order (7.31) reduces to

$$\mathrm{e}^{2\mathrm{i}I} = -\mathrm{i}. \tag{7.33}$$

However, it appears that $I = O(\beta^{-\frac{1}{2}})$, and so there is a contradiction, which can only be avoided if

$$\int_1^{r_0} [\hat{A}_0 \log r - \mathrm{i}]^{\frac{1}{2}} \mathrm{d}r = 0, \tag{7.34}$$

or
$$\int_0^{\mathrm{i}/A_0} \mathrm{e}^{-\rho} \rho^{\frac{1}{2}} \mathrm{d}\rho = 0, \tag{7.35}$$

or
$$\gamma(\tfrac{3}{2}, \mathrm{i}/A_0) = 0, \tag{7.36}$$

where $\gamma(z_1, z_2)$ represents the incomplete gamma function. This represents an eigenvalue problem for $\hat{A}_0$, which was solved numerically using a combination of trapezoidal quadrature and Newton iteration; results for the first few $\hat{A}_0$ are shown in table 2. Note that there appear to be many values (probably an infinite number),

$\hat{A}_0$
$0.1408+0.2262\times10^{-1}\mathrm{i}$
$0.7455\times10^{-1}+0.7991\times10^{-2}\mathrm{i}$
$0.5076\times10^{-1}+0.4181\times10^{-2}\mathrm{i}$
$0.3848\times10^{-1}+0.2603\times10^{-2}\mathrm{i}$
$0.3099\times10^{-1}+0.1790\times10^{-2}\mathrm{i}$
$0.2594\times10^{-1}+0.1313\times10^{-2}\mathrm{i}$
$0.2231\times10^{-1}+0.1007\times10^{-2}\mathrm{i}$
$0.1952\times10^{-1}+0.7996\times10^{-3}\mathrm{i}$
$0.1742\times10^{-1}+0.6514\times10^{-3}\mathrm{i}$
$0.1571\times10^{-1}+0.5418\times10^{-3}\mathrm{i}$

TABLE 2. Values of $\hat{A}_0$

although these seem to be concentrated within a finite annular region in the complex $\hat{A}_0$-plane. As the order increased, the values become very close to neighbouring values, and the computation became exceedingly difficult; however, with increasing order the values of $\hat{A}_0$ do seem to be approaching a finite value (indeed the author was unable to find any solution for $|\hat{A}_0| \lesssim 0.098$). Note too that it is easy to show using integration by parts that there are no solutions to (7.35) as $|\hat{A}_0| \to \infty$, whilst using the asymptotic expansion for the incomplete gamma function (Abramowitz & Stegun 1964) it is also possible to show that no solutions exist as $|\hat{A}_0| \to 0$ either; this then confirms our statement about the values of A_0 being confined to an annular region in complex $\hat{A}_0$-space.

Note also that both $\hat{A}_0$ and complex conjugate $\{\hat{A}_0\}$ are roots of (7.35); however, the latter family of solutions may be disregarded since in all cases we require $\hat{A}_0$ to possess a positive real part.

The second family of solutions for A_0 occurs when $A_0 = O(\beta^{-\frac{3}{2}})$. In this case, from (7.27), $|r_0-1| \ll 1$, and indeed the wall $(r=1)$ lies inside the transition layer. Consequently, we are unable to use (7.31), but must consider the transition layer in detail (although this is quite a routine task).

Suppose
$$A_0 = \beta^{-\frac{3}{2}}\tilde{A}_0, \tag{7.37}$$
where $\tilde{A}_0 = O(1)$. Then defining
$$\hat{\zeta} = (r-1)\beta^{-\frac{1}{2}}, \tag{7.38}$$
to leading order (5.10) reduces to
$$\psi_{00\hat{\zeta}\hat{\zeta}\hat{\zeta}} + \psi_{00\hat{\zeta}}(\tilde{A}_0\hat{\zeta}-\mathrm{i}) - \tilde{A}_0\psi_{00} = 0. \tag{7.39}$$
Writing
$$\hat{\zeta} = (-\tilde{A}_0)^{-\frac{1}{3}}\sigma + \mathrm{i}/\tilde{A}_0, \tag{7.40}$$
and differentiating (7.39) with respect to $\hat{\zeta}$, yields
$$\psi_{00\sigma\sigma\sigma\sigma} - \sigma\psi_{00\sigma\sigma} = 0. \tag{7.41}$$
The required solution (that is not exponentially large as $\sigma \to \infty$) is
$$\psi_{0\sigma\sigma} = D\,\mathrm{Ai}\,(\sigma) \tag{7.42}$$
(where D is independent of σ).

The implementation of the boundary conditions on $\hat{\zeta} = 0$ requires $\psi_{00\hat{\zeta}\hat{\zeta}\hat{\zeta}}(\hat{\zeta}=0) = 0$, and so
$$\mathrm{Ai}'\left(\frac{-\mathrm{i}(-\tilde{A}_0)^{\frac{1}{3}}}{\tilde{A}_0}\right) = 0. \tag{7.43}$$

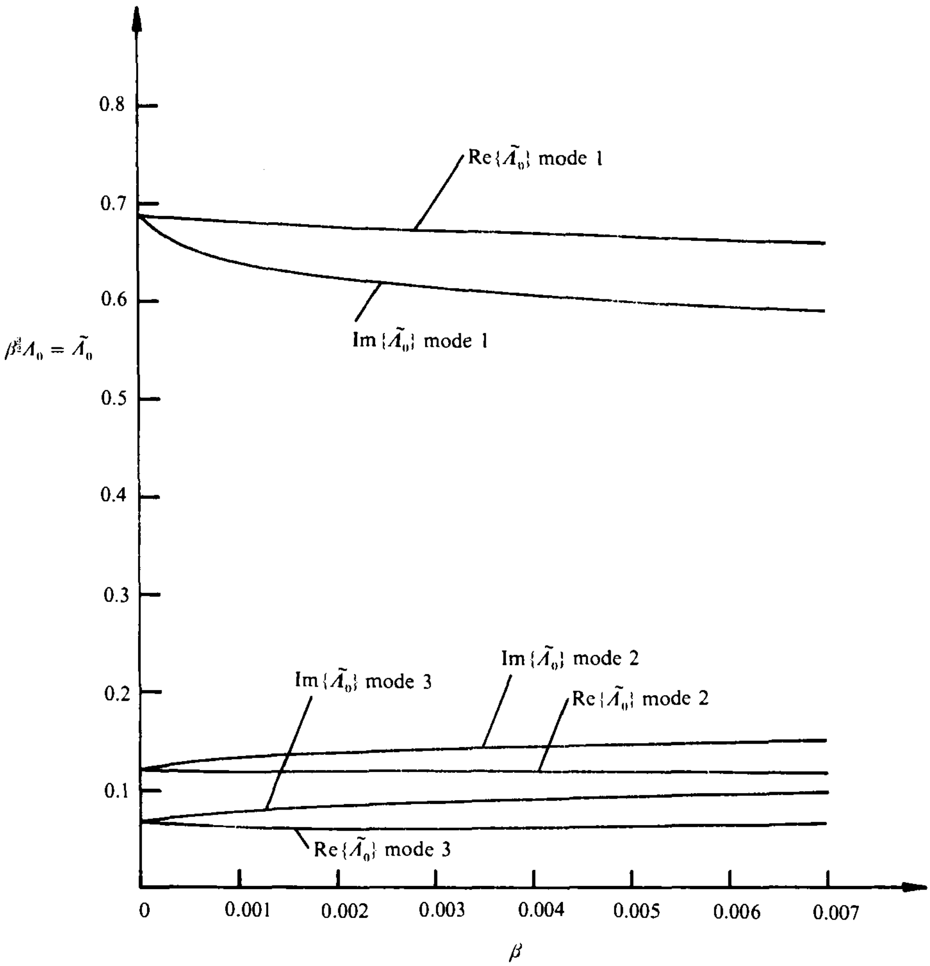

FIGURE 8. Variation of $\beta^{\frac{3}{2}}A_0$ with β, first three modes.

Since the zeros of the Airy function and its derivative are confined exclusively to the negative real axis, then

$$\mathrm{Ai}'(-\zeta_n) = 0 \quad (n = 1, 2, 3, \ldots), \tag{7.44}$$

where the ζ_n are real and positive and tabulated by Abramowitz & Stegun (1964). Consequently

$$\tilde{A}_0 = \frac{1+\mathrm{i}}{\sqrt{2}\zeta_n^{\frac{3}{2}}} + O(\beta^{\frac{1}{2}}), \tag{7.45}$$

where the appropriate roots have been chosen to ensure boundedness of the Airy function.

It is interesting (although, in some ways not too surprising) that (7.45) is identical to the corresponding expression found in the analogous planar study (Lam & Rott 1960; Ackerberg & Phillips 1972; Goldstein 1983), although of course the corresponding $f(Z)$ is quite different in the present case.

As a check on the numerical results as $\beta \to 0$, figure 8 shows the variation of $\beta^{\frac{3}{2}}A_0$ with β (first three modes). It is very clear that these results approach those given by

(7.45) as $\beta \to 0$. The $O(\beta^{-1})$ family of results refer to higher modes, and thus it is not realistically possible to compare our numerical results with this family.

In the following section we draw some conclusions from this work.

8. Conclusions

In this paper the effect of small-amplitude free-stream oscillations on an otherwise steady boundary layer in an axisymmetric body has been investigated. Particular attention has been focused on the far-downstream eigenvalues and eigensolutions. As noted in §1, in the case of the planar problem, two distinct families of eigensolutions have been presented, namely those originally considered by Lam & Rott (1960) and those considered by Brown & Stewartson (1973*a*, *b*), with the former family having decay rates that decrease with increasing order, whilst the latter family have decay rates that increase with increasing order. In the present study, eigenvalues appear to occur with decreasing decay rate with increasing order. However, some of the asymptotic work in §7 (in particular that relevant to $\beta \to \infty$, with $\Lambda_0 = O(\beta^{-1})$) does strongly suggest that a finite value of Λ_0 is being approached with increasing order. Indeed, the author was unable to obtain a consistent asymptotic solution to (5.10) for $\beta = O(1)$, $\Lambda_0 \to 0$, again suggesting the finite limit of Λ_0 with increasing order. This, in some ways may be regarded as a rather more satisfactory state of affairs than that found with the Lam & Rott (1960) eigensolutions, which have decay rates that become diminishingly small with increasing order (although see our comments, attributed to Goldstein *et al.* 1983, in §1). Further the $\beta \to 0$ work of §7 does suggest that all modes possess the same decay rate in this limit up to at least second order.

However, it may well be that the planar work of Brown & Stewartson (1973*a*, *b*) could perhaps be extended to include the effects of curvature, to yield a further (perhaps related) family of eigensolutions. A further interesting study would be an investigation of the far-downstream evolution of the eigensolutions. Just as in the planar case, these all become increasingly oscillatory far downstream, and will, as a consequence, ultimately cease to be valid approximations to the Navier–Stokes equations. This will lead, presumably, to the formation of unstable Tollmien–Schlichting waves, in a manner analogous to that described by Goldstein (1983) in the planar case.

However, there are a number of (other) important differences between the planar and the axisymmetric eigensolutions and eigenvalues. Most importantly the downstream (i.e. axial) behaviour of these eigensolutions (described by $f(Z)$) which is quite different in the two cases, in the axisymmetric case being given by (5.6) whilst in the planar Blasius case

$$f(x) = \mathrm{e}^{-\Lambda x^{\frac{3}{2}}} x^p, \tag{8.1}$$

as shown by Goldstein (1983) (where x is the streamwise coordinate). Note that if the basic flow were of the form $\Psi = x^m F(\eta)$, with $\eta = y/x^m$, y being the transverse boundary-layer variable, then using arguments similar to those in this paper,

$$f(Z) = x^p \exp\{-\Lambda x^{2n-m+1}\} \quad \text{for} \quad 2n - m + 1 > 0. \tag{8.2}$$

The computations presented in this paper were carried out with computer time provided by the University of Manchester and SERC Grant no. GR/E/25702. This research was also supported by the National Aeronautics and Space Administration under NASA contract no. NAS1-18605 while the author was in residence at the

Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA23665. The author wishes to thank the referees for some very helpful comments on an earlier version of this paper, including pointing out equation (7.22).

Appendix

Consider terms

$$O\{h(Z)\,Z^{-\frac{1}{2}A_0 Z}\,\mathrm{e}^{\frac{1}{2}A_0 Z}\,Z^{p-1}\,(\log Z)^{q-1}\}$$

in (5.2) to enable us to determine the value of q.

It is found, after some algebra, that

$$\begin{aligned}L\{\psi_{11}\} = p\left\{\frac{\Psi'_{00}\psi'_{01}}{r}+\frac{\Psi'_{01}\psi'_{00}}{r}-\psi_{00}\left[\frac{\Psi''_{01}}{r}-\frac{\Psi'_{01)}}{r_2}\right]-\psi_{01}\left[\frac{\Psi''_{00}}{r}-\frac{\Psi'_{00}}{r^2}\right]\right\}\\ +q\left\{\frac{\psi'_{00}\Psi'_{00}}{2r}-\tfrac{1}{2}\psi_{00}\left[\frac{\Psi''_{00}}{r}-\frac{\Psi'_{00}}{r^2}\right]\right\}+A_1\left\{\frac{\Psi'_{00}\psi'_{10}}{r}-\frac{\Psi'_{10}\psi'_{00}}{r}\right.\\ \left.+\psi_0\left[\frac{\Psi''_{10}}{r}-\frac{\Psi'_{10}}{r^2}\right]+\psi_{10}\left[\frac{\Psi''_{00}}{r}-\frac{\Psi'_{00}}{r^2}\right]\right\}+A_0\left\{-\frac{\psi'_{10}\Psi'_{01}}{r}-\frac{\Psi'_{10}\psi'_{01}}{r}\right.\\ -\frac{\Psi'_{11}\psi'_{00}}{r}+\psi_{00}\left[\frac{\Psi''_{11}}{r}-\frac{\Psi'_{11}}{r^2}\right]+\psi_{01}\left[\frac{\Psi''_{10}}{r}-\frac{\Psi'_{10}}{r^2}\right]\\ \left.+\psi_{10}\left[\frac{\Psi''_{01}}{r}-\frac{\Psi'_{01}}{r^2}\right]\right\}-\frac{1}{2r}\psi'_{00}\Psi'_{00}+\tfrac{1}{2}\Psi_{00}\left[\frac{\psi''_{00}}{r}-\frac{\psi'_{00}}{r^2}\right].\end{aligned}\qquad (\mathrm{A}\ 1)$$

Recalling the expressions already obtained for p, A_1, ψ_{01}, ψ_{10}, (A 1) leads to the following (slightly simplified) equation:

$$\begin{aligned}L(\psi_{11}\} = Q(r)\,\{A_0K_{10}A_{01}+\tfrac{1}{2}q+A_0K_{01}A_{10}-(A_{10}+K_{10})A_0\}\\ -\frac{\Psi'_{11}\psi'_{00}}{r}+\psi_{00}\left[\frac{\Psi''_{11}}{r}-\frac{\Psi'_{11}}{r^2}\right]+\frac{1}{2r}\psi'_0\Psi'_{00}+\tfrac{1}{2}\Psi_{00}\left[\frac{\psi''_{00}}{r}-\frac{\psi'_{00}}{r^2}\right],\end{aligned}\qquad (\mathrm{A}\ 2)$$

where

$$Q(r) = \frac{\Psi'_{00}\psi'_{00}}{r}-\psi_{00}\left[\frac{\Psi''_{00}}{r}-\frac{\Psi'_{00}}{r^2}\right].\qquad (\mathrm{A}\ 3)$$

It is quite clear that $\psi_{11} = O(r^2(\log r)^2)$ as $r\rightarrow\infty$. To simplify arguments later, we write

$$\psi_{11} = \psi^{\mathrm{I}}_{11}(r)+q\psi^{\mathrm{II}}_{11}(r),\qquad (\mathrm{A}\ 4)$$

where $\psi^{\mathrm{II}}_{11}(r)$ is any regular function which has the following behaviour as $r\rightarrow\infty$:

$$\psi^{\mathrm{II}}_{11}(r) = -\frac{A_{00}}{\beta A_0^2}+o(1).\qquad (\mathrm{A}\ 5)$$

the governing equation for ψ^{I}_{11} may then be written in the form

$$L\{\psi^{\mathrm{I}}_{11}\} = R_3(r)+qR_4(r).\qquad (\mathrm{A}\ 6)$$

where

$$R_3(r) = Q(r)\{A_0 K_{10} A_{01} + A_0 K_{01} A_{10} - (A_{10}+K_{10}) A_0\} - \frac{\Psi'_{11}\psi'_{00}}{r}$$
$$+\psi_{00}\left[\frac{\Psi'''_{11}}{r} - \frac{\Psi'_{11}}{r^2}\right] + \frac{1}{2r}\psi'_0\,\Psi'_{00} + \tfrac{1}{2}\Psi_{00}\left[\frac{\psi''_{00}}{r} - \frac{\psi'_0}{r^2}\right],$$

and

$$R_4(r) = -\tfrac{1}{2}Q(r) - L\{\psi^{\mathrm{II}}_{11}\}. \tag{A 7}$$

The boundary conditions that must be applied to this system are

$$\left.\begin{aligned} &\psi^{\mathrm{I}}_{11}(1) = -\psi^{\mathrm{II}}_{11}(1), \qquad \psi^{\mathrm{I}\prime}_{11}(1) = -\psi^{\mathrm{II}\prime}_{11}(1) \\ &\psi^{\mathrm{I}}_{11} \to 0 \quad \text{as} \quad r\to\infty. \end{aligned}\right\} \tag{A 8}$$

and that

The value of q is then determined by the condition that a solution to this system exists.

Consider now the (complex conjugate of the) adjoint to the system (5.10), denoted by $\psi^+(r)$, and determined from

$$\psi^{+'''} + \frac{\psi^{+''}}{r} + \psi^{+'}[A_0 \log r - \mathrm{i}/\beta - 1/r^2] + 2A_0/r\,\psi^+ = 0. \tag{A 9}$$

subject to the boundary conditions

$$\psi^+(r=1) = 0, \tag{A 10}$$

$$\psi^+ = o((\log r)^{-2}) \quad \text{as} \quad r\to\infty. \tag{A 11}$$

If we now further suppose, as we are quite at liberty to so do (although this simplifies, but is not crucial for our arguments), that

$$\psi^{\mathrm{II}}_{11}(1) = \psi^{\mathrm{II}\prime}_{11}(1) = 0, \tag{A 12}$$

then

$$q = -\int_1^\infty R_3(r)\,\psi^+(r)\,\mathrm{d}r \bigg/ \int_1^\infty R_4(r)\,\psi^+(r)\,\mathrm{d}r \tag{A 13}$$

(Hartman 1964, for example).

This (at least in principle) determines, or provides a means of determining the index of the logarithmic term multiplying $f(Z)$. At this stage it would also appear to be legitimate to set the function $h(Z)$ in (5.6) equal to a constant, although categorical determination of this point seems difficult because of the algebraic complexity in extending the analysis to higher order.

REFERENCES

ABRAMOWITZ, M. & STEGUN, I. A. 1964 *Handbook of Mathematical Functions, with Formulas, Graphs and Mathematical Tables*. Washington DC: US Government Printing Office.

ACKERBERG, R. C. & PHILLIPS, J. H. 1972 The unsteady laminar boundary layer on a semi-infinite plate plate due to small fluctuations in the magnitude of the freestream velocity. *J. Fluid Mech.* **51**, 137.

BROWN, S. N. & STEWARTSON, K. 1973*a* On the propagation of disturbances in a laminar boundary layer I. *Proc. Camb. Phil. Soc.* **73**, 493.

BROWN, S. N. & STEWARTSON, K. 1973*b* On the propagation of disturbances in a laminar boundary layer II. *Proc. Camb. Phil. Soc.* **73**, 503.

BUSH, W. B. 1976 Axial incompressible viscous flow past a slender body of revolution. *Rocky Mount. J. Maths* **6**, 527.

Duck, P. W. 1984 The effect of a surface discontinuity on an axisymmetric boundary layer. *Q. J. Mech. Appl. Maths* **37**, 57.

Duck, P. W. 1989 A numerical method for treating time-periodic boundary layers. *J. Fluid Mech.* **204**, 544.

Duck, P. W. & Bodonyi, R. J. 1986 The wall jet on an axisymmetric body. *Q. J. Mech. Appl. Maths* **34**, 407.

Duck, P. W. & Hall, P. 1989 On the interaction of axisymmetric Tollmien–Schlichting waves in supersonic flow. *Q. J. Mech. Appl. Maths* **42**, 115 (also *ICASE Rep.* 88–10).

Glauert, M. B. & Lighthill, M. J. 1955 The axisymmetric boundary layer on a long thin cylinder. *Proc. R. Soc. Lond.* A **230**, 188.

Goldstein, M. E. 1983 The evolution of Tollmien–Schlichting waves near a leading edge. *J. Fluid Mech.* **127**, 59.

Goldstein, M. E., Sockol, P. M. & Sanz, J. 1983 The evolution of Tollmien–Schlichting waves near a leading edge. Part 2. Numerical determination of amplitudes. *J. Fluid Mech.* **129**, 443.

Hartman, R. J. 1964 *Ordinary Differential Equations.* John Wiley.

Keller, H. B. & Cebeci, T. 1971 Accurate numerical methods for boundary-layer flows. Part 1. Two-dimensional laminar flows. In *Proc. 2nd Intl Conf. on Numerical Methods in Fluid Dynamics* (ed. M. Holt). Lecture Notes in Physics, vol. 8. Springer.

Lam, S. H. & Rott, N. 1960 Theory of linearised time-dependent boundary layers. *Cornell University GSAE Rep.* AFOSR. TN-60-1100.

Lighthill, M. J. 1954 The response of laminar skin friction and heat transfer to fluctuations in the stream velocity. *Proc. R. Soc. Lond.* A **224**, 1.

Pedley, T. J. 1972 Two-dimensional boundary layers in a free stream which oscillate without reversing. *J. Fluid Mech.* **55**, 359.

Phillips, J. H. & Ackerberg, R. C. 1973 A numerical method for integrating the unsteady boundary-layer equations when there are regions of backflow. *J. Fluid Mech.* **58**, 561.

Rott, N. & Rosenweig, M. L. 1980 On the response of the laminar boundary layer to small fluctuations of the free-stream velocity. *J. Aero. Sci.* **27**, 741.

Seban, R. A. & Bond, R. 1951 Skin friction and heat transfer characteristics of a laminar boundary layer on a cylinder in axial incompressible flow. *J. Aero. Sci.* **10**, 171.

Stewartson, K. 1955 The asymptotic boundary layer on a circular cylinder in axial incompressible flow. *Q. Appl. Maths* **13**, 113.